A Graph-theory approach for analyzing the effects of ordering on ILU preconditionning

S. Doi, A. Lichnewsky

UNITÉ DE RECHERCHE
INRIA-ROCQUENCOURT

Institut National
de Recherche
en Informatique
et en Automatique

Domaine de Voluceau
Rocquencourt
B.P. 105
78153 Le Chesnay Cedex
France
Tél.: (1) 39 63 55 11

Rapports de Recherche

N° 1452

Programme 1
Architectures parallèles, Bases de données, Réseaux et Systèmes distribués

A GRAPH-THEORY APPROACH FOR ANALYZING THE EFFECTS OF ORDERING ON ILU PRECONDITIONING

Shun DOI
Alain LICHNEWSKY

Juin 1991

* R R . 1 4 5 2 *

A GRAPH-THEORY APPROACH FOR ANALYZING THE EFFECTS OF ORDERING ON ILU PRECONDITIONING*

ANALYSE DES EFFETS DE LA NUMÉROTATION SUR LE PRÉCONDITIONNEMENT ILU PAR LA THÉORIE DES GRAPHES

SHUN DOI [†] and ALAIN LICHNEWSKY [‡]

*This work was carried out while S. Doi was visiting INRIA (Institut National de Recherche en Informatique et en Automatique, Rocquencourt, BP105 - 78153, Le Chesnay Cedex, France).

[†] C&C Information Technology Research Laboratories, NEC Corporation, 4-1-1, Miyazaki, Miyamae-ku, Kawasaki, 216 Japan. E-mail: doi@ibl.cl.nec.co.jp.

[‡] Advanced Computer Research Institute, CNIT-BP 240, 92053 Paris la Defense, France. On leave from INRIA, Rocquencourt.

Abstract

Studied here are the effects of ordering on the speed of convergence to solutions for ILU preconditioned iterative methods. A graph-theory approach is applied to an element-by-element analysis of the remainder matrix $R = M - A$, where M is the ILU factorization of a matrix A (which is based on a given ordering). A digraph representation of ordering is used as a unique expression for a class of algebraically equivalent ILU factorizations. The properties of full compatibility and partial compatibility are introduced and defined in terms of the digraph structure. The results of analytical and empirical studies are presented, and the effects of ordering of unknowns are seen to be fairly well explained by the properties of full compatibility and partial compatibility.
Keywords: Linear systems, ILU factorization, convection-diffusion, preconditioner, ordering, convergence, parallel computation.

Résumé

Nous étudions l'influence de l'ordre des inconnues sur la vitesse de convergence des méthodes itératives de type ILU préconditionnées. Notre approche utilise la théorie des graphes pour analyser les coefficients non nuls de la matrice "reste" $R = M - A$, où M est la factorisation ILU d'une matrice A selon un ordre donné. Une classe de factorisations ILU algébriquement équivalentes est représentée par un digraphe caractérisant l'ordre des inconnues correspondant. Dans ce formalisme, nous introduisons les propriétés de compatibilité partielle et compatibilité totale. Les résultats d'études analytiques et empiriques sont présentés; nous montrons que l'influence de la renumérotation est très bien caractérisée par les propriétés de compatibilité totale et de compatibilité partielle.

1. Introduction. This paper is concerned with preconditioned iterative methods for solving the large sparse nonsymmetric linear system

$$Au = b, \tag{1.1}$$

which arises from the finite difference discretization of convection-diffusion equations. A preconditioned iterative method for (1.1) is defined as a basic iterative method, such as BCG [11], CGS [20] or CGSTAB [24], for the preconditioned system

$$M^{-1}Au = M^{-1}b,$$

where M is the preconditioner. The most popular and successful preconditioners include the no-fill-in incomplete LU (ILU) point factorizations [14, 16, 21]. Let D be the diagonal of matrix A, while its strictly lower and strictly upper triangles are represented, respectively, as L and U. In this case, the no-fill-in ILU factorization for A may be defined as

$$M = (\Delta + \hat{L})\Delta^{-1}(\Delta + \hat{U}), \tag{1.2}$$

where the diagonal matrix Δ is defined by the equation

$$\Delta + \mathrm{diag}(\hat{L}\Delta^{-1}\hat{U}) = D, \tag{1.3}$$

and the strictly lower and strictly upper triangular matrices $\hat{L}$ and $\hat{U}$, whose non-zero structure is equal to that of L and U, are determined such that

$$\text{off-diag}(A) = \text{corresponding}(M). \tag{1.4}$$

In Eq. (1.3), $\mathrm{diag}(\hat{L}\Delta^{-1}\hat{U})$ is the diagonal part of $\hat{L}\Delta^{-1}\hat{U}$. Equation (1.4) implies that the non-zero off-diagonal elements of A should be equal to the corresponding elements of M [21].

If A has Property-A [26], e.g., if A arises from the 2D 5-point or 3D 7-point finite differences, Eq. (1.4) results in $L = \hat{L}$ and $U = \hat{U}$. Equations (1.2) and (1.3) may then be simplified as

$$M = (\Delta + L)\Delta^{-1}(\Delta + U) \ \text{ with } \ \Delta + \mathrm{diag}(L\Delta^{-1}U) = D. \tag{1.5}$$

On vector and/or parallel computers, forward and backward substitutions by $(\Delta + L)$ and $(\Delta + U)$ are the most time consuming parts of the overall process for preconditioned iterative methods. One way to attain parallelism in forward-backward solving is to reorder the linear system (1.1). Let P be a permutation matrix. The linear system (1.1) will then be equivalent to the reordered system

$$\tilde{A}\tilde{u} = \tilde{b}, \tag{1.6}$$

where $\tilde{A} = PAP^T$, $\tilde{u} = Pu$ and $\tilde{b} = Pb$. Let $\tilde{D}$, $\tilde{L}$ and $\tilde{U}$ be, respectively, the diagonal, the strictly lower triangle and the strictly upper triangle of $\tilde{A}$. The ILU factorization for $\tilde{A}$,

$$\tilde{M} = (\tilde{\Delta} + \tilde{L})\tilde{\Delta}^{-1}(\tilde{\Delta} + \tilde{U}) \ \text{ with } \ \tilde{\Delta} + \mathrm{diag}(\tilde{L}\tilde{\Delta}^{-1}\tilde{U}) = \tilde{D},$$

may possibly have better parallelism than Eq. (1.5). However, these factorizations, M and $\tilde{M}$, are not always algebraically equivalent (see §2.1), and their convergence characteristics are not always the same. Thus, there is potentially the problem of the need for tradeoff between parallelism and convergence.

Lichnewsky *et al.* [10, 15] used nested dissection ordering [12] for the vector multiprocessor implementation of the ICCG method. Poole and Ortega [19] and Oyanagi [18] introduced multicolor ordering to vectorize the ICCG method. Duff and Meurant [6] have shown, by intensive numerical experiments, that the influence of ordering on convergence in the ICCG method is very great, especially for hard problems (anisotropy, discontinuity, etc. in coefficients). The Red-Black (RB) ordered ICCG method, for example, required up to 6 times the number of iterations required by the Row (natural) ordered ICCG method. They have also shown that a method with a "smaller" remainder matrix $R = M - A$ tends to give better convergence for the ICCG method. They, as well as Poole and Ortega, have reported that a larger aspect ratio in grid-sizes makes for poorer convergence in the RB/ICCG method. Aschcraft and Grimes [1] have pointed out that poor convergence of the RB/ICCG can potentially offset the speedup gained by vectorization. Tradeoff between parallelism and convergence have also been reported in [2, 8, 9].

The object of this paper is to analyze, using a graph-theory approach, the tradeoff problem encountered in ILU preconditioned iterative methods. Section 2 discusses aspects of graph theory fundamental to the study. In Section 3, the properties of full and partial compatibility are introduced and defined in terms of a digraph (directed graph) structure of ordering. It is shown that, under certain assumed conditions, ILU factorization becomes asymptotically equivalent to complete LU factorization if the underlying ordering is compatible. The remainder matrices of ILU factorizations are analyzed for some specific orderings of matrices which arise from the 5-point upwind finite differencing of 2D convection-diffusion equations. It is observed that, if the ordering is compatible, the remainder terms tend to decrease with any change in PDE parameters, but not otherwise. Numerical experiments are carried out in Section 4, and these show good agreement with the results of the remainder matrix analysis in Section 3.

2. Graph-theory. In this section, we discuss and define the tools borrowed from graph-theory and used in subsequent sections to treat aspects of ordering in ILU factorizations. The theorems and algorithms developed here are applicable to sparse matrices in general.

2.1. Equivalence of ILU factorizations. Digraphs are used here to represent sparse matrices in general.

DEFINITION 1 (DIGRAPH). *A labeled digraph $G(B) = (V, E)$ for an $N \times N$ sparse matrix $B = (b_{ij})$ is defined by a node (or vertex) set $V = \{p_1, \ldots, p_N\}$ and an edge set $E = \{e(i,j) | b_{ji} \neq 0\}$. Digraph $G(B)$ is said to be "unlabeled" when N nodes are not distinguished by name $(p_1, \ldots, p_N)$. Digraph $G(B)$ is said to be "induced" by B.*

Nodes p_i and p_j are the *initial* and *terminal* nodes of an edge $e(i, j)$, respectively. In Definition 1, the direction of the edge is defined in such a way that it expresses the influence of input x_i on output y_j in the matrix-vector multiplication $y = Bx$, where $x = [x_1, \ldots, x_N]^T$ and $y = [y_1, \ldots, y_N]^T$.

The digraph defined here is the *converse* of those defined in [7, 13].

DEFINITION 2 (EQUALITY OF DIGRAPHS). *Two digraphs $G(B)$ and $G(\tilde{B})$, induced by B and $\tilde{B}$, are equal $(G(B) = G(\tilde{B}))$ if a permutation matrix P exists such that $\tilde{B} = PBP^T$.*

Let us assume, for example, a finite difference grid or finite element grid composed of N nodes and some number of *undirected* edges which connect adjacent nodes. By adding directions to each of these edges, we obtain an unlabeled digraph, from which we may deduce possible orderings. (A digraph represents an ordering when its nodes are numbered in such a way that the initial node of each edge is labeled with a number smaller than that of its terminal node.)

Obviously, then, two digraphs are equal if they are induced by any orderings which are deduced from a single digraph.

Let us assume $A = L + D + U$ to be structurally symmetric, i.e., $e(i,j) \in E(L) \Leftrightarrow e(j,i) \in E(U)$, and define the equivalence of ILU factorizations as follows:

DEFINITION 3 (EQUIVALENCE OF ILU FACTORIZATIONS). *Assume that M and $\tilde{M}$ are the ILU factorizations of A and $\tilde{A}$, and that $\tilde{A} = PAP^T$, where P is a permutation matrix. If $\tilde{M} = PMP^T$, then the ILU factorizations M and $\tilde{M}$ are equivalent. Their underlying orderings are also said to be equivalent with respect to the ILU factorization.*

Clearly, the preconditioned CG-type methods for Eqs. (1.1) and (1.6) are algebraically equal if their orderings are equivalent. The digraph representation has fundamental importance since it gives a unique representation for the class of equivalent ILU factorizations, as described below.

THEOREM 2.1. *Assume that the ILU factorizations*

$$M = (\Delta + L)\Delta^{-1}(\Delta + U) \quad \text{with} \quad \Delta + diag(L\Delta^{-1}U) = D, \tag{2.1}$$

$$\tilde{M} = (\tilde{\Delta} + \tilde{L})\tilde{\Delta}^{-1}(\tilde{\Delta} + \tilde{U}) \quad \text{with} \quad \tilde{\Delta} + diag(\tilde{L}\tilde{\Delta}^{-1}\tilde{U}) = \tilde{D}, \tag{2.2}$$

exist respectively for A and $\tilde{A} = PAP^T$, where P is a permutation matrix. In this situation, they are equivalent if and only if $G(L) = G(\tilde{L})$.

The proof for this theorem may be accomplished separately for the factorization and substitution processes.

Proof of equivalence in the factorization process. Here, we see that the diagonal matrices Δ and $\tilde{\Delta}$ defined by Eqs. (2.1) and (2.2) satisfy $\tilde{\Delta} = P\Delta P^T$ if and only if $G(L) = G(\tilde{L})$. We may give proof for the permutation of two succeeding elements, and the other permutations can be accomplished by repeating this process.

Let $A = (a_{ij})$ and $\Delta = \text{diag}[d_1 \ldots d_n]$. The ith element of Δ may then be broken down as

$$d_i = a_{ii} - \sum_{k:e(k,i)\in E(L)} a_{ik} d_k^{-1} a_{ki},$$

where $E(L)$ is the edge set of $G(L)$. The summation is taken over k such that $e(k,i) \in E(L)$. For the jth element,

$$d_j = a_{jj} - \sum_{k:e(k,j)\in E(L)} a_{jk} d_k^{-1} a_{kj}.$$

Let $j = i + 1$, and P be the permutation of i and j. The ith and jth elements, denoted by d'_i and d'_j, for the permuted system may then be written as

$$d'_i = a'_{ii} - \sum_{k:e(k,i)\in E(L')} a'_{ik} d'^{-1}_k a'_{ki} = d_j + a_{ji} d_i^{-1} a_{ij}, \tag{2.3}$$

$$d'_j = a'_{jj} - \sum_{k:e(k,j)\in E(L')} a'_{jk} d'^{-1}_k a'_{kj}. = d_i - a_{ji} d_i^{-1} a_{ij}, \tag{2.4}$$

where $L' = PLP^T$.

Assume $G(L) = G(L')$. Then $e(i,j) \notin E(L)$, implying that $a_{ji}d_i^{-1}a_{ij} = 0$. From Eqs. (2.3) and (2.4), it follows that $d_i' = d_j$ and $d_j' = d_i$. The succeeding elements of Δ are not affected by this interchange either. Hence, $G(L) = G(L') \Rightarrow \Delta' = P\Delta P^T$.

Assume $G(L) \neq G(L')$. Then $e(i,j) \in E(L)$. Since $a_{ji}d_i^{-1}a_{ij} \neq 0$, $d_i' \neq d_j$ and $d_j' \neq d_i$. Hence, $\Delta' \neq P\Delta P^T$. That is, $G(L) \neq G(L') \Rightarrow \Delta' \neq P\Delta P^T$. Combining these results, it follows that $\tilde{\Delta} = P\Delta P^T \Leftrightarrow G(L) = G(\tilde{L})$.

Let $i+1 < j$, i.e., k exists such that $i < k < j$. Assume $e(k,j) \notin E(L)$ for any $k(i < k < j)$. Then j and its proceeding element can be permuted without causing any difference in the algebraic relation, as shown above. By repeating this process, it becomes equivalent to the case of $j = i + 1$.

Assume k exists such that $k(i < k < j)$ and $e(k,j) \in E(L)$. If l does not exist such that $e(i,l)$ and $e(l,k) \in E(L)$, then i and j can be interchanged by first interchanging i and k (as shown above), and then i and j, so that k comes first, and then j and i. If l exists such that $e(i,l)$ and $e(l,k) \in E(L)$, there is no way to interchange i and j without changing the algebraic relationship and the digraph.

In more general terms, if j is not *reachable* (in graph terminology [13]) from i, then the digraph does not change, and i and j can be interchanged without causing any difference in the algebraic relationship. Otherwise, the digraph and the algebraic relationship will change if i and j are interchanged.

Proof of equivalence in the substitution process. We see here that Eqs. (2.1) and (2.2) satisfy $\tilde{M} = PMP^T$ if and only if $G(L) = G(\tilde{L})$. Let us assume that $\tilde{\Delta} = P\Delta P^T$. Consider the solution of $Mx = z$ by the forward-backward substitutions

$$y = \Delta^{-1}(z - Ly), \tag{2.5}$$

$$x = y - \Delta^{-1}Ux. \tag{2.6}$$

The forward process (2.5) may be written for i and j as

$$y_i = d_i^{-1}(z_i - \sum_{k:e(k,i)\in E(L)} a_{ik}y_k), \tag{2.7}$$

$$y_j = d_j^{-1}(z_j - \sum_{k:e(k,j)\in E(L)} a_{jk}y_k). \tag{2.8}$$

The backward process (2.6) may be written as

$$x_j = y_j - d_j^{-1}(\sum_{k:e(j,k)\in E(L)} a_{jk}x_k), \tag{2.9}$$

$$x_i = y_i - d_i^{-1}(\sum_{k:e(i,k)\in E(L)} a_{ik}x_k). \tag{2.10}$$

Assume $j = i+1$. Let P be the permutation of i and j. If $G(L) = G(L')$ where $L' = PLP^T$, then $e(i,j) \notin E(L)$. Since the summation in Eq. (2.8) does not include y_i, Eqs. (2.7) and (2.8) are interchangeable. Similarly, Eqs. (2.9) and (2.10) are interchangeable. Hence, $M' = PMP^T$. It also follows that $G(L) = G(L') \Rightarrow M' = PMP^T$.

If $G(L) \neq G(L')$, then $e(i,j) \in E(L)$. In this case, the summation in Eq. (2.8) will include y_i, so Eqs. (2.7) and (2.8) will not be interchangeable, i.e., $L' \neq PLP^T$. It is also be true for the backward process. Hence, $G(L) \neq G(L') \Rightarrow M' \neq PMP^T$. Thus, $\tilde{M} = PMP^T \Leftrightarrow G(L) = G(\tilde{L})$. The rest of the proof for $i+1 < j$ is essentially the same as the proof for the factorization process. □

Theorem 2.1 implies that the digraph gives a unique representation for a class of equivalent orderings. In the following discussion, the term *ordering* will often be used to refer to a *class of equivalent orderings.*

2.2. Structure of remainder matrices. The remainder matrix $R = (r_{ij}) = M - A$ may be rewritten as $R = \text{off-diag}(L\Delta^{-1}U)$, where off-diag$(\cdot)$ is the off-diagonal part of the matrix. r_{ij} $(i \neq j)$ may then be broken down as

$$\begin{aligned} r_{ij} &= \sum_{k:k<i,k<j} a_{ik}d_k^{-1}a_{kj} \\ &= \sum_{k:e(k,i)\in E(L),e(k,j)\in E(L)} a_{ik}d_k^{-1}a_{kj}. \end{aligned}$$

This implies that

$$r_{ij}\begin{cases} = 0 & \text{if } \{k|e(k,i) \in E(L), e(k,j) \in E(L)\} = \emptyset \\ \neq 0 & \text{if } \{k|e(k,i) \in E(L), e(k,j) \in E(L)\} \neq \emptyset \end{cases} \tag{2.11}$$

Let $S(R[i])$ be the node set $\{p_j|e(j,i) \in E(R)\}$, where $E(R)$ is the edge set for the remainder matrix R. From Eq. (2.11), $S(R[i])$ may then be rewritten, by using $E(L)$, as

$$S(R[i]) = \{p_j|\forall k, e(k,i) \in E(L) \cap e(k,j) \in E(L) \cap j \neq i\}. \tag{2.12}$$

Expression (2.12) gives the following algorithm for listing up the elements of $S(R[i])$:

Algorithm RNS.

Step 1: List up all k such that $e(k,i) \in E(L)$,

Step 2: List up all $j(\neq i)$ such that $e(k,j) \in E(L)$ for all k listed in Step 1.

The set of j then gives $S(R[i])$.

This graph-theory approach to discussing the structure of R will serve as a background for the discussion of various orderings in §3.

2.3. Parallelism in ILU factorizations. Let us introduce here a few graph notations related to parallel computation.

Definition 4 (Source node). *A node p_i in a digraph $G = (V,E)$ is a source node if $\{e(k,i)|p_k \in V, e(k,i) \in E\} = \emptyset$ (an empty set).*

Consider the linear system $(\Delta + L)y = z$, where Δ and L are diagonal and strictly lower triangular matrices and y is an unknown vector. This system may be solved by forward substitution (2.5), which will break down into Eq. (2.7). Since Eq. (2.7) only requires unknown y_k's such that $e(k,i) \in E(L)$, the substitution for unknowns corresponding to source nodes can be started immediately, and conducted individually for each individual source node.

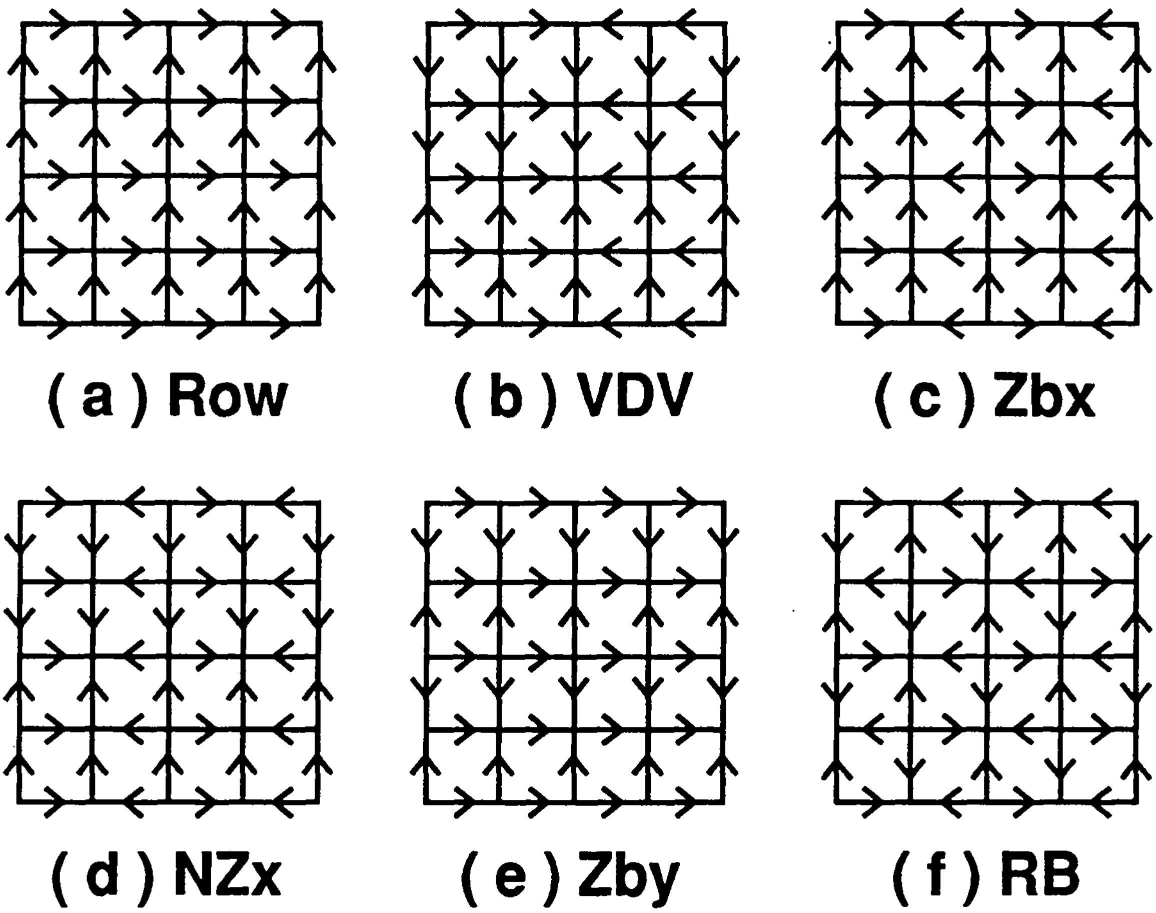

FIG. 1. *Digraph representation of orderings discussed in this paper.*

Assume that an unlimited number of arithmetic processors exist for which each executes one arithmetic operation in a single time unit, and that each can communicate with any other processor with no time delay. The forward substitution may then be accomplished in this system in t time units, where t is the length of the longest path in $G(L)$. This allows us to define the degree of parallelism for a digraph.

DEFINITION 5 (PARALLELISM DEGREE). *Let t be the length of the longest path in $G(L)$, and let N be the number of nodes. The degree of parallelism D_p for $G(L)$ may then be defined by*

$$D_p = N/t.$$

Consider a rectangular 2D 5-point finite difference grid. The most popular orderings are Row and Red-Black (RB) orderings, whose digraph representations are given in Figures 1 (a) and (f). Their respective degrees of parallelism are $n^2/(2n-1) \approx n/2$ and $n^2/2$, where n is the grid-size for both the x and y directions.

Figure 1 (b) is a digraph representation of a VDV ordering [1]. The VDV ordering graph can

[1] The ILU factorization based on VDV ordering is algebraically equivalent to the Twisted factorization (based on the Row ordering) introduced by Van der Vorst [23].

TABLE 1
Numerical Experiments by Duff and Meurant [6].

Ordering	Equivalence	Compatibility	Cosistency	Number of iterations DM1	DM2	DM3	DM4
row	row	f.c.	c.	23	15	68	9
cm	row	f.c.	c.	23	16	68	9
rcm	–	f.c.	c.	23	15	68	9
block	row	f.c.	c.	23	16	68	9
mind	–	icm.	icn.	39	24	108	48
rb	rb	icm.	c.	38	23	107	47
altd	rb	icm.	c.	38	23	107	47
zebra	–	p.c.(x)	c.	28	27	71	9
nest	–	icm.	icn.	25	20	83	26
diss1	–	p.c.(y)	c.	23	18	75	21
diss2	–	p.c.(y)	c.	24	18	84	26
spiral	–	f.c.	icn.	23	15	69	9
4col	–	icm.	c.	33	27	103	34
vdv1	vdv1	f.c.	c.	20	15	68	9
vdv2	vdv1	f.c.	c.	20	15	67	9
ujac	–	icm.	icn.	28	20	86	34
loc	–	icm.	icn.	23	16	70	17

f.c. – fully compatible, c. – consistent,
p.c. – partially compatible, icn. – inconsistent,
icm. – incompatible.

cm – Cuthill McKee, diss1 – One way dissection(1 level),
rcm – Reverse Cuthill McKee, diss2 – One way dissection(2 level),
block – Block Cuthill McKee, 4col – 4-color,
mind – Minimum degree, vdv1 – Van der Vorst(1),
rb – Red black, vdv2 – Van der Vorst(2),
altd – Alternating direction, ujac – Union Jack,
nest – Nested dissection, loc – Localized row/column.

Other conditions are as follows:
Example DM1: (Laplacian) 5-point finite difference. $k_x = k_y = 1$, $\sigma = 0$.
Example DM2: (Laplacian) 9-point finite difference. $k_x = k_y = 1$, $\sigma = 0$.
Example DM3: (Discontinuous coefficient) 5-point finite difference. $\sigma = 0.01$.

Sub-domain	k_x	k_y
$(0, 0.5] \times (0, 0.5]$	1	1
$(0.5, 1) \times (0, 0.5]$	100	1
$(0, 0.5] \times (0.5, 1)$	1	100
$(0.5, 1) \times (0.5, 1)$	100	100

Example DM4: (Anisotropic coefficient) 5-point finite difference. $k_x = 100$, $k_y = 1$, $\sigma = 0$.
The numbers of iterations for these examples are shown in Table 1, which also gives equivalence, compatibility and consistency [26]. Examples DM1 and DM4 correspond to the analytical results

Further, with respect to the spectral radius of $M^{-1}R$ and the condition number of $M^{-1}A$, the following theorem will hold if we assume both A and M to be symmetric, positive and definite.

THEOREM 3.2. *Assume that $A = M - R$ is a regular splitting, where A is a symmetric positive definite M-matrix, and M is symmetric, positive and definite. The spectral radius ρ (< 1) of $M^{-1}R$ and the condition number ν of $M^{-1}A$ will then satisfy the inequality*

$$1/(1-\rho) \leq \nu \leq (1+\rho)/(1-\rho). \tag{3.1}$$

Proof. If A and M are symmetric, positive and definite, then $\nu = \lambda_{max}/\lambda_{min}$, where λ_{max} and λ_{min} (> 0) are the maximum and minimum eigenvalues of $M^{-1}A$. Let μ_{max} and μ_{min} be the maximum and minimum eigenvalues of $M^{-1}R$. If $A = M - R$ is regular, then $\rho = \mu_{max}$ [25]. Since $M^{-1}A = I - M^{-1}R$,

$$\lambda_{min} = 1 - \mu_{max} = 1 - \rho, \quad \lambda_{max} = 1 - \mu_{min}.$$

Since $|\mu_{min}| \leq |\mu_{max}| = \rho$,

$$1 \leq \lambda_{max} \leq 1 + \rho.$$

Consequently, Eq. (3.1) is reached. ∎

For example, the condition number is in the range $10^4 \leq \nu \leq 2 \cdot 10^4$ when $\rho = 0.9999$, and is reduced to the range $10^3 \leq \nu \leq 2 \cdot 10^3$ when $\rho = 0.999$. Further,

$$\lim_{\rho \to 0} \nu = 1, \quad \lim_{\rho \to 1} \nu = \infty.$$

Hence, it is expected that the "smaller" the R, the smaller the condition number ν. A collection of basic theorems regarding condition numbers can also be seen in Ortega [17].

3.2. The effects of ordering under limiting conditions. In order to discuss the effects of ordering under limiting conditions, let us use a model problem, the convection-diffusion equation

$$(k_x u'_x)'_x + (k_y u'_y)'_y + v_x u'_x + v_y u'_y = g(x, y) \text{ on } \Omega \tag{3.2}$$

with Dirichlet boundary conditions on $\partial\Omega$. $(\cdot)'_x$ and $(\cdot)'_y$ denote partial differentiations. For the sake of simplicity, domain Ω is restricted to rectangular form, and $k_x(> 0)$, $k_y(> 0)$, v_x and v_y are assumed to be constant (though it is not strictly necessary to maintain these restrictions for the following theorems to hold true). Let us examine here how the element-by-element size of R, which is based on individual orderings, is affected by change in the model parameter set (k_x, k_y, v_x, v_y).

Let $Au = b$ be the linear system obtained by discretizing Eq. (3.2), and denote the left-hand side of the system at node (i, j) by

$$(Au)_{ij} = bu_{ij-1} + cu_{i-1j} + u_{ij} + eu_{i+1j} + fu_{ij+1}, \tag{3.3}$$

where

$$\begin{aligned} c &= \{k_x + (|v_x| - v_x)/2\}/D, \quad b = \{k_y + (|v_y| - v_y)/2\}/D, \\ e &= \{k_x + (|v_x| + v_x)/2\}/D, \quad f = \{k_y + (|v_y| + v_y)/2\}/D, \\ D &= -2(k_x + k_y) - |v_x| - |v_y|. \end{aligned} \tag{3.4}$$

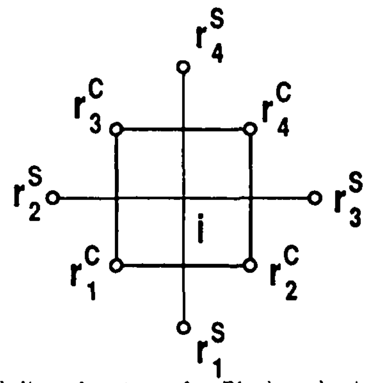

FIG. 3. *Local digraph pattern for Black nodes in RB ordering.*

The coefficients (3.4) are obtained when the convection and diffusion terms are discretized by 1st-order upwind and 2nd-order centered differences, where grid-size h is fixed at one. Note that A is a diagonally dominant M-matrix. The effect of variable h can be included in the above discretization if the PDE parameters are modified accordingly.

Equation (3.3) may be rewritten by the stencil expression as

$$(A)_{ij} = \begin{bmatrix} & f & \\ c & 1 & e \\ & b & \end{bmatrix}.$$

Then, by applying *Algorithm RNS*(§2.2) to the possible digraph patterns of L around p_{ij}, it follows that the stencil of R has the form

$$(R)_{ij} = \begin{bmatrix} & & r_{ij}^{S4} & & \\ & r_{ij}^{C3} & \cdot & r_{ij}^{C4} & \\ r_{ij}^{S2} & \cdot & \cdot & \cdot & r_{ij}^{S3} \\ & r_{ij}^{C1} & \cdot & r_{ij}^{C2} & \\ & & r_{ij}^{S1} & & \end{bmatrix}, \tag{3.5}$$

and that it has no other non-zero elements. The superscripts "S" and "C" indicate "Star" and "Cross" positions.

Consider, for example, the node p_{ij} in the local digraph shown in Fig. 3, which is for Black nodes in the RB ordering. In Step 1, the set $\{p_{ij-1}, p_{i-1j}, p_{i+1j}, p_{ij+1}\}$ is obtained as "k". By applying Step 2 to p_{ij-1}, as "k", the set $\{p_{ij-2}, p_{i-1j-1}, p_{i+1j-1}\}$ is obtained as "j". Similarly, it turns out that $S(R[(i,j)]) = \{p_{ij-2}, p_{i-1j-1}, p_{i+1j-1}, p_{i-2j}, p_{i+2j}, p_{i-1j+1}, p_{i+1j+1}, p_{ij+2}\}$, whose stencil expression is Eq. (3.5).

Let $R = R^S + R^C$, where R^S and R^C are composed of those star and cross remainder terms, respectively. They may then be rewritten as

$$(R^S)_{ij} = \begin{bmatrix} & & \gamma_{ij+1}^{yy} f^2/d_{ij+1} & & \\ & & \cdot & & \\ \gamma_{i-1j}^{xx} c^2/d_{i-1j} & \cdot & \cdot & \cdot & \gamma_{i+1j}^{xx} e^2/d_{i+1j} \\ & & \cdot & & \\ & & \gamma_{ij-1}^{yy} b^2/d_{ij-1} & & \end{bmatrix}, \tag{3.6}$$

$$(3.7) \qquad (R^C)_{ij} = \begin{bmatrix} \gamma^{x-y-}_{ij+1} cf/d_{ij+1} & \cdot & \gamma^{x+y-}_{ij+1} ef/d_{ij+1} \\ +\gamma^{x+y+}_{i-1j} cf/d_{i-1j} & & +\gamma^{x-y+}_{i+1j} ef/d_{i+1j} \\ \cdot & \cdot & \cdot \\ \gamma^{x-y+}_{ij-1} bc/d_{ij-1} & & \gamma^{x+y+}_{ij-1} be/d_{ij-1} \\ +\gamma^{x+y-}_{i-1j} bc/d_{i-1j} & \cdot & +\gamma^{x-y-}_{i+1j} be/d_{i+1j} \end{bmatrix},$$

where d_{ij} are the Δ elements, defined by Eq. (1.5). γ_{ij} are mask operators which are either 0 or 1. They are determined from the local graph structure around p_{ij}. For example, $\gamma^{xx}_{i-1j} = 1$ if $e(p_{i-1j}, p_{ij})$ and $e(p_{i-1j}, p_{i-2j}) \in E(L)$, and $= 0$ otherwise; $\gamma^{x-y-}_{ij+1} = 1$ if $e(p_{ij+1}, p_{ij})$ and $e(p_{ij+1}, p_{i-1j+1}) \in E(L)$, and $= 0$ otherwise; etc.

An interesting observation may be made here from Eqs. (3.6) and (3.7). The numerators in R^S are the squares of coefficients in A, e.g., c^2, whereas those in R^C are multiples of two different coefficients, e.g., bc, each of which corresponds to the different directions x and y. Intuitively, if one (or more) of PDE parameters changes, the effects will be amplified in R^S, but not in R^C. Moreover, non-zero elements in R^S, e.g., r^{S2}_{ij}, appear if and only if the digraph has the form $p_{i-2j} \leftarrow p_{i-1j} \rightarrow p_{ij}$, etc. (see Eq. (2.11)).

To permit a more precise discussion, let us introduce and define a few new terms. For individual edge pairs along one direction, three different patterns exists, i.e.,

$p_{i-1} \rightarrow p_i \leftarrow p_{i+1}$: I/I elementary graph,
$p_{i-1} \rightarrow p_i \rightarrow p_{i+1}$: I/O elementary graph,
$p_{i-1} \leftarrow p_i \rightarrow p_{i+1}$: O/O elementary graph.

Here, "I" and "O" are abbreviations for "Input" and "Output". Another possible pattern $p_{i-1} \leftarrow p_i \leftarrow p_{i+1}$ is categorized as the I/O elementary graph.

DEFINITION 6 (INCOMPATIBLE NODE). *A node p_i is called incompatible if one or both of its elementary graph is O/O. "x-incompatible" nodes are those in which the elementary graph in the x-direction is O/O, and "y-incompatible" nodes are those in which the y-direction elementary graph is O/O.*

DEFINITION 7 (FULLY COMPATIBLE ORDERING). *An ordering is referred to as "fully compatible" if the ordering digraph has no incompatible nodes.*

DEFINITION 8 (PARTIALLY COMPATIBLE ORDERING). *An ordering is referred to as "partially compatible" if the ordering digraph has incompatible nodes only for one direction, and "partially x- (or y-) compatible" if the digraph has only y- (or x-) incompatible nodes.*

For example, the Row and VDV orderings in Fig. 1 are fully compatible. The Zebra and ND-Zebra orderings are partially compatible; Zbx and NZx are partially y-compatible and Zby is partially x-compatible. The RB ordering is incompatible.

We may then derive the following theorem for expressing, within certain limits placed on PDE parameters, the relationship between levels of compatibility and the speed of convergence of ILU factorization.

THEOREM 3.3. *Let A be a matrix arising from the 5-point upwind finite differencing (see Eqs. (3.3), (3.4)) of convection-diffusion equation (3.2). Assume that ILU factorization $M = (\Delta + L)\Delta^{-1}(\Delta + U)$ exists for A, where $\delta > 0$ exists such that $\Delta - \delta I \geq 0$. Then $\lim_{k_x \to \infty} R = \lim_{|v_x| \to \infty} R = \lim_{k_x, |v_x| \to \infty} R = 0$ if and only if the ordering is partially x-compatible. The same relation also holds for the y direction.*

Proof. Assume that the ordering is x-compatible. Definition 8 then tells us that $r^{S2}_{ij} = r^{S3}_{ij} = 0$ for any i,j (see Eq. (3.5)). Since the other terms, r^{S1}_{ij}, r^{S4}_{ij} and r^{Cl}_{ij} ($l = 1, \ldots, 4$), may be non-zero, it may be shown that, for any small ϵ, k'_x (and v'_x) exist such that $r^{S1}_{ij} \leq \epsilon$ for any $k_x \geq k'_x$, etc.

Let us r_{ij}^{S1}, where k_x goes to infinity and v_x, k_y and v_y are fixed. For given ϵ, one can find k_x' such that $r_{ij}^{S1} \le \epsilon$ for any $k_x \ge k_x'$. In fact,

$$\begin{aligned} r_{ij}^{S1} = b^2/d_{ij-1} &= \{k_y + (|v_y| - v_y)/2\}^2/[\{2(k_x + k_y) + |v_x| + |v_y|\}^2 d_{ij-1}] \\ &\le \{k_y + (|v_y| - v_y)/2\}^2/[\{2(k_x + k_y) + |v_x| + |v_y|\}^2 \delta], \end{aligned}$$

where $\gamma_{ij+1}^{yy} = 1$ is assumed. Hence, $r_{ij}^{S1} \le \epsilon$ for any k_x such that

$$k_x \ge k_x' = (\epsilon\delta)^{-1/2}\{k_y + (|v_y| - v_y)/2\}/2 - k_y - (|v_x| + |v_y|)/2.$$

Similarly, one can find k_x' such that $r_{ij}^{S4} \le \epsilon$ and $r_{ij}^{Cl} \le \epsilon$ ($l = 1, \ldots, 4$) for any $k_x \ge k_x'$. The same observation may be made for the case of $|v_x| \to \infty$, as well as for both k_x and $|v_x| \to \infty$.

Let us next assume that node p_{ij} is incompatible. Definition 6 then tells us that $r_{i+1j}^{S2} \ne 0$, and $r_{i-1j}^{S3} \ne 0$.

$$r_{i+1j}^{S2} = c^2/d_{ij} = \{k_x + (|v_x| - v_x)/2\}^2/\{2(k_x + k_y) + |v_x| + |v_y|\}^2/d_{ij}.$$

Consequently,

$$\lim_{k_x \to \infty} r_{i+1j}^{S2} = 0.25 d_{ij}^{-1} \ne 0, \quad \lim_{v_x \to -\infty} r_{i+1j}^{S2} = d_{ij}^{-1} \ne 0, \quad \text{etc.}$$

Similarly,

$$\lim_{k_x \to \infty} r_{i-1j}^{S3} = 0.25 d_{ij}^{-1} \ne 0, \quad \lim_{v_x \to \infty} r_{i-1j}^{S3} = d_{ij}^{-1} \ne 0, \quad \text{etc.}$$

This completes the proof for partial x-compatibility. The proof for y-compatibility may be similarly performed. ∎

COROLLARY 3.1. *Under the assumption of Theorem 3.3,*

$$\lim_{k_x \to \infty} R = \lim_{|v_x| \to \infty} R = \lim_{k_x, |v_x| \to \infty} R = \lim_{k_y \to \infty} R = \lim_{|v_y| \to \infty} R = \lim_{k_y, |v_y| \to \infty} R = 0$$

if and only if the ordering is fully compatible.

Proof. This may be seen to follow directly from Theorem 3.3. ∎

This theorem and its corollary imply that, under certain limiting conditions, ILU factorization becomes equivalent to complete LU factorization if the ordering is fully or partially compatible and if factorization satisfies $\Delta - \delta I \ge 0$. As the aspect ratio for grid sizes Δx and Δy increases, PDE parameters approach infinity. Hence, where the underlying ordering is fully or partially compatible, we may expect good convergence with the ILU preconditioned iterative method, particularly in anisotropic problems.

Meijerink and Van der Vorst have shown that a unique ILU factorization exists if A is an M-matrix [16]. In their proof, there is no indication of the existence of a δ such that $\Delta - \delta I \ge 0$. Hence, in a variant of Theorem 3.3 for an M-matrix, the "δ" assumption ($\delta > 0$ exists such that $\Delta - \delta I \ge 0$) will be still necessary. In practical programs, to ensure the accuracy of succeeding computations, it is preferable to modify M so that δ will be positive. That is, if $d_{ij} \le \delta$ is detected while in the factorization process, the program will then replace d_{ij} with a positive value and continue the factorization. Kershaw's variant operates in this manner [14]. We may expect ILU factorization to be close to complete if, for PDE parameters approaching infinity, such replacement occurs only rarely and compatibility is either full or partial.

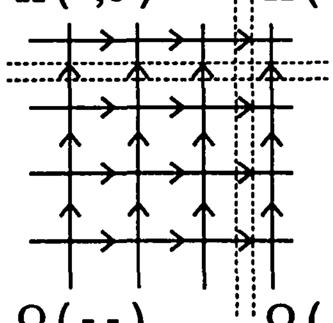

FIG. 4. *A quarter of the VDV ordering graph to be analyzed.*

Let us next consider the relationship between compatibility and parallelism. Forward substitution can be started in parallel from source nodes (see Definition 4). Assume that a source node resides in domain Ω (exclude nodes on $\partial\Omega$). The source node will then be incompatible for both the x and y directions. Hence, any ordering which has source nodes inside of the domain will be incompatible. Similarly, any source node on the boundary will not be fully compatible. Thus, we may derive the following corollaries:

COROLLARY 3.2. *A fully compatible ordering can have source nodes only at corners.*

COROLLARY 3.3. *A partially compatible ordering can have source nodes on parallel edges (including corners).*

The VDV ordering in Fig. 1, which has four source nodes, is a case of Corollary 3.2. Clearly, no other fully compatible ordering exists which has more source nodes in 2D rectangular grids.

3.3. Element-by-element analysis of R for several orderings. This subsection give more quantitative results than §3.2 for certain specific orderings. Figure 1 shows the orderings studied in this subsection. The diagonal element of Δ, d_{ij}, is generally defined in terms of certain recursive relationships, e.g., as in the Row ordering,

$$d_{ij} = 1 - b_{ij}f_{ij-1}/d_{ij-1} - c_{ij}e_{i-1j}/d_{i-1j}.$$

In [3], the remainder terms were analyzed by assuming that a sequence $\{d_{ij}\}$ has a limit d (as $i,j \to \infty$), determined by

$$d = 1 - bf/d - ce/d. \tag{3.8}$$

In some of the examples, this assumption was justified for sufficiently small i,j.

Let us consider the linear system of the form defined by (3.3) and (3.4) under this assumption (see *Remarks*). The difference from [3] is that different limits must be posed for different local graphs. Consider the VDV ordering (see Fig. 2). The digraphs of the four dissected domains are isomorphic. Hence, only a quarter of the full domain needs to be analyzed (Fig. 4). The quarter portion can then be decomposed into four parts, the portion of which will depend on the local graph structure. Let $\Omega(x,y)$ denote the domain. Generic terms "+" and "−" may be used instead of specific x or y values, e.g., $\Omega(0,-)$ denotes the negative part of the x-axis. The four portions may then be written as $\Omega(0,0)$, $\Omega(-,0)$, $\Omega(0,-)$ and $\Omega(-,-)$. Assume that a sequence $\{d_{ij}\}$ in each sub-domain converges to an individual limit. Note that the limit for $\Omega(-,-)$ is

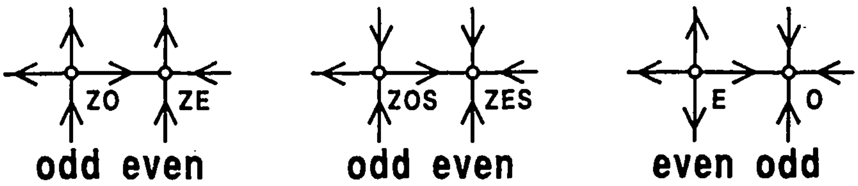

FIG. 5. *Spatial expressions of "d".*

equal to that for the Row ordering. Let d_R, d_{VH}, d_{VV} and d_{VM} be limits corresponding to these sub-domains. They may then be written as

$$
\begin{aligned}
d_R &= \{1+\sqrt{1-4(bf+ce)}\}/2, \\
d_{VH} &= \{1-2bf/d_R+\sqrt{(1-2bf/d_R)^2-4ce}\}/2, \\
d_{VV} &= \{1-2ce/d_R+\sqrt{(1-2ce/d_R)^2-4bf}\}/2, \\
d_{VM} &= 1-2bf/d_{VV}-2ce/d_{VH}.
\end{aligned}
\tag{3.9}
$$

Similarly, the following limits are obtained for the Zbx, NZx and RB orderings (Fig. 5).

$$
\begin{aligned}
d_{ZO} &= \{1+\sqrt{1-4bf}\}/2, \\
d_{ZE} &= \{1-2ce/d_{ZO}+\sqrt{(1-2ce/d_{ZO})^2-4bf}\}/2, \\
d_{ZOS} &= 1-2bf/d_{ZO}, \\
d_{ZES} &= 1-2bf/d_{ZE}-2ce/d_{ZOS}, \\
d_E &= 1, \\
d_O &= 1-2bf-2ce.
\end{aligned}
\tag{3.10}
$$

Consequently, the non-zero remainder terms may be estimated as follows:

Row ordering.

$$r_R^{C2} = be/d_R, \quad r_R^{C3} = cf/d_R.$$

VDV ordering (Horizontal Separator).

$$r_{VH}^{C2} = be/d_R, \quad r_{VH}^{C4} = ef/d_R.$$

Zbx ordering.

$$
\begin{aligned}
r_{ZO}^{C1} = bc/d_{ZO}, \quad r_{ZO}^{C2} = be/d_{ZO}, \quad r_{ZE}^{C3} &= cf/d_{ZO}, \quad r_{ZE}^{C4} = ef/d_{ZO}, \\
r_{ZE}^{S2} = c^2/d_{ZO}, \quad r_{ZE}^{S3} &= e^2/d_{ZO}.
\end{aligned}
$$

NZx ordering (Separator).

$$
\begin{aligned}
r_{ZOS}^{C1} = bc/d_{ZO}, \quad r_{ZOS}^{C2} = be/d_{ZO}, \quad r_{ZOS}^{C3} &= cf/d_{ZO}, \quad r_{ZOS}^{C4} = ef/d_{ZO}, \\
r_{ZES}^{S2} = c^2/d_{ZOS}, \quad r_{ZES}^{S3} &= e^2/d_{ZOS}.
\end{aligned}
$$

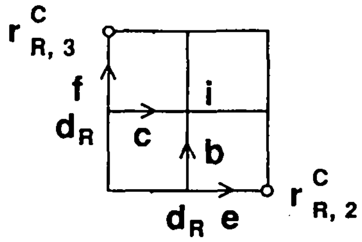

FIG. 6. *Row ordering remainder structure.*

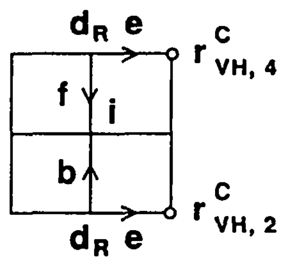

FIG. 7. *VDV ordering (Horizontal Separator) remainder structure.*

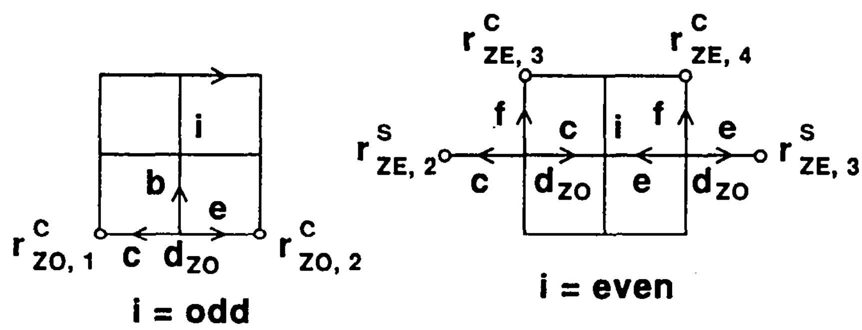

FIG. 8. *Zbx ordering remainder structure.*

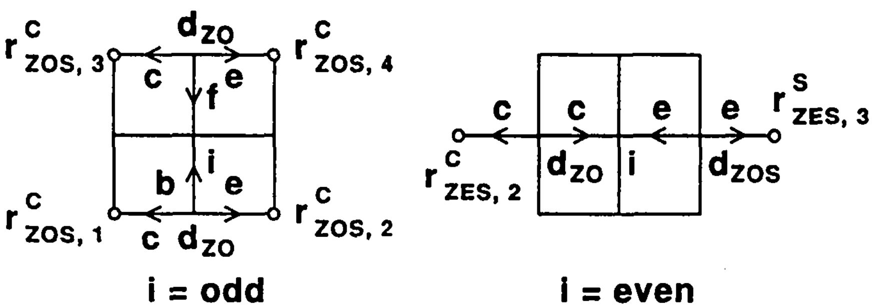

FIG. 9. *NZx ordering (Separator) remainder structure.*

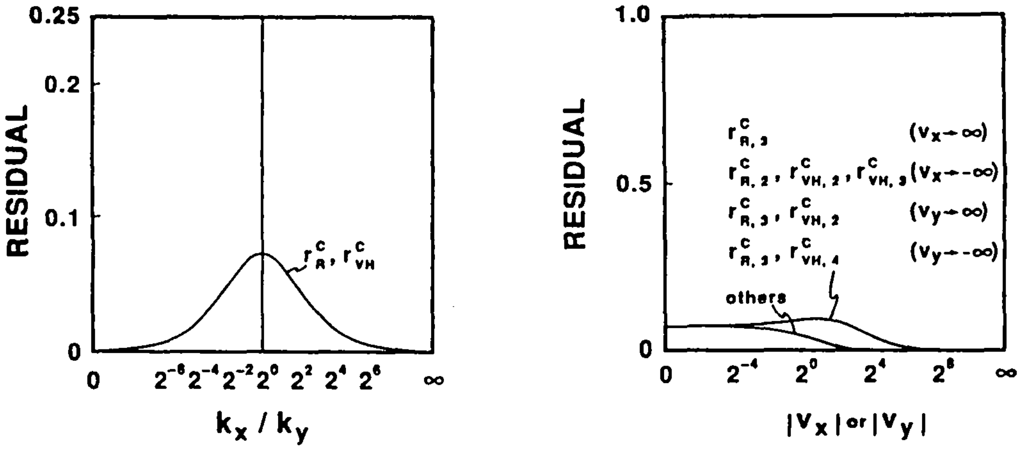

FIG. 10. *Remainders for fully compatible orderings.*

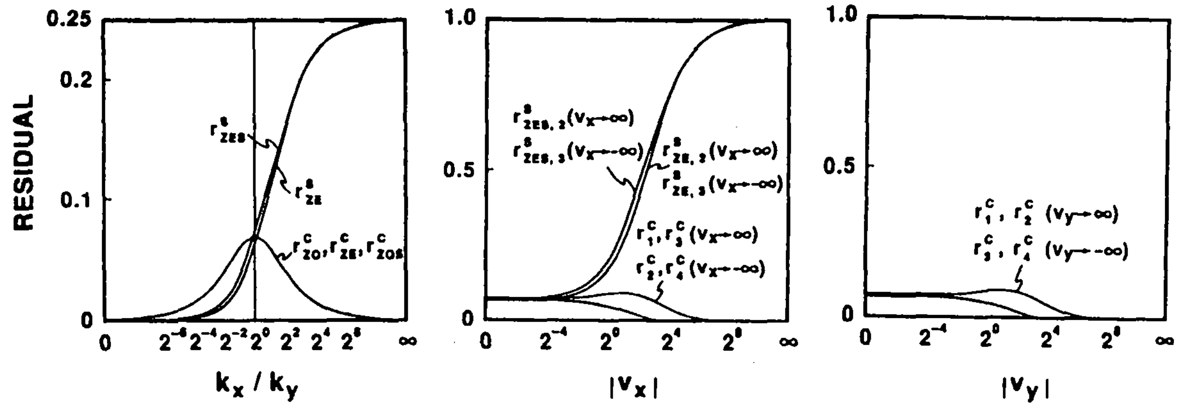

FIG. 11. *Remainders for partially y compatible orderings.*

RB ordering.

$$
\begin{aligned}
r_O^{C1} &= 2bc, \quad r_O^{C2} = 2be, \quad r_O^{C3} &= 2cf, \quad r_O^{C4} = 2ef, \\
r_O^{S1} &= b^2, \quad r_O^{S2} = c^2, \quad r_O^{S3} &= e^2, \quad r_O^{S4} = f^2.
\end{aligned}
$$

Figures 6–9 show the relation between these remainder elements and nodes. The remainder structure for RB ordering is shown in Fig. 2. Since the VDV ordering vertical separator is isomorphic to the horizontal separator, it is not analyzed. The VDV ordering center node and the RB ordering even nodes have no remainder elements.

Figures 10–12 show how remainder terms change in size with respect to PDE parameters. In Figs. 10–12 (a), k_x and k_y change, while $v_x = v_y = 0$. In Figs. 10-12 (b) and (c), either v_x or v_y changes, while $k_x = k_y = 1$. Figure 10 shows the results for Row and VDV fully compatible orderings, Fig. 11 for Zbx and NZx partially y compatible orderings, and Fig. 12 for RB incompatible ordering.

From these results, several conclusions may be reached. The size of remainder terms coincides with the results found in §3.2, when the parameter goes to infinity. The size of remainder terms monotonically decreases when the anisotropy of diffusion parameters increases, i.e., $k_x/k_y \to 0$ or ∞. It is also monotonic for changes in convection intensity, except for an element which has a peak around v_x or $v_y \simeq 2$ in any ordering.

We may recall that the existence of non-zero remainder terms (R^S) is determined by the local graph structure (2.12), while the size of individual elements in Δ (each of which corresponds

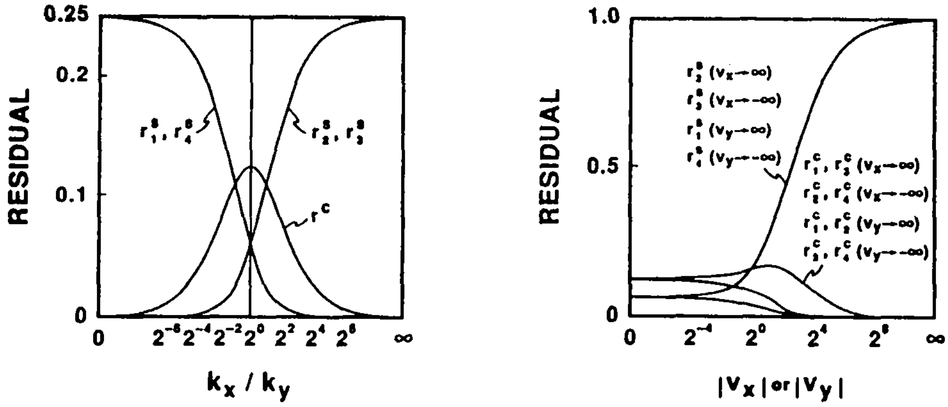

FIG. 12. *Remainders for incompatible orderings.*

to a node in the graph) is dependent upon the structure of the graph over a wider range. This range is that which includes all nodes for which the node in question is reachable (see the proof of Theorem 2.1). However, the results seen in Figs. 10–12 imply that the size of non-zero R^S is more strongly influenced by their position than by the underlying global ordering. That is, if two different orderings have non-zero remainder terms at the same position, the size of these terms may be expected to be very close.

Remarks. The analysis in here is based on the assumption that a sequence $\{d_{ij}\}$ converges to a limit d for fairly small i, j, and in order to test the validity of this assumption, we need to know whether sequences really behave in this way [2].

For the sequence $\{d_{ij}\}$, let $d_{ij} = d(1 + \delta_{ij})$, where d is defined by Eqs. (3.9) and (3.10). Let us calculate here how fast the δ_{ij} (for the various "d"s found in the remainder terms: d_R, d_{ZO}, d_{ZOS}) converge to zero. The following table shows how δ_{ij} for d_R behaves for a set of PDE parameter sets.

			(i,j)		
(k_x, k_y, v_x, v_y)	(1,1)	(2,2)	(3,3)	(4,4)	(5,5)
$(1,1,0,0)$	0.14645	0.01311	0.00125	0.00012	0.00001
$(100,1,0,0)$	0.42999	0.18488	0.10530	0.06705	0.04524
$(1,1,1,0)$	0.13944	0.01263	0.00130	0.00014	0.00002
$(1,1,100,0)$	0.00952	0.00009	0.00000	0.00000	0.00000

Even in the worst case $(100,1,0,0)$, δ_{ij} is less than 5%, for $i, j \geq 5$. For the Zebra ordering, the convergence was much faster. It should also be noted that $\delta_{ij} > 0$. Hence, it can be concluded that the estimated remainder terms in §3.3 give fairly nice upperbounds.

4. Numerical experiments. In this section, we examine how well the results of numerical experiments agree with the theoretically derived results of §3.

4.1. Duff and Meurant's experiments. This subsection refers to the results presented by Duff and Meurant [6]. Their model equations are of the form

$$-(k_x u'_x)'_x - (k_y u'_y)'_y + \sigma u = g \text{ on } \Omega = (0,1) \times (0,1).$$

[2] For Row ordering, Van der Vorst has shown that the limit d_R exists and that convergence is fairly fast (see Ch. 3, Appendix A [22]).

TABLE 1
Numerical Experiments by Duff and Meurant [6].

Order-ing	Equiva-lence	Compati-bility	Cosist-ency	Number of iterations DM1	DM2	DM3	DM4
row	row	f.c.	c.	23	15	68	9
cm	row	f.c.	c.	23	16	68	9
rcm	–	f.c.	c.	23	15	68	9
block	row	f.c.	c.	23	16	68	9
mind	–	icm.	icn.	39	24	108	48
rb	rb	icm.	c.	38	23	107	47
altd	rb	icm.	c.	38	23	107	47
zebra	–	p.c.(x)	c.	28	27	71	9
nest	–	icm.	icn.	25	20	83	26
diss1	–	p.c.(y)	c.	23	18	75	21
diss2	–	p.c.(y)	c.	24	18	84	26
spiral	–	f.c.	icn.	23	15	69	9
4col	–	icm.	c.	33	27	103	34
vdv1	vdv1	f.c.	c.	20	15	68	9
vdv2	vdv1	f.c.	c.	20	15	67	9
ujac	–	icm.	icn.	28	20	86	34
loc	–	icm.	icn.	23	16	70	17

f.c. – fully compatible, c. – consistent,
p.c. – partially compatible, icn. – inconsistent,
icm. – incompatible.

cm – Cuthill McKee, diss1 – One way dissection(1 level),
rcm – Reverse Cuthill McKee, diss2 – One way dissection(2 level),
block – Block Cuthill McKee, 4col – 4-color,
mind – Minimum degree, vdv1 – Van der Vorst(1),
rb – Red black, vdv2 – Van der Vorst(2),
altd – Alternating direction, ujac – Union Jack,
nest – Nested dissection, loc – Localized row/column.

Other conditions are as follows:
Example DM1: (Laplacian) 5-point finite difference. $k_x = k_y = 1$, $\sigma = 0$.
Example DM2: (Laplacian) 9-point finite difference. $k_x = k_y = 1$, $\sigma = 0$.
Example DM3: (Discontinuous coefficient) 5-point finite difference. $\sigma = 0.01$.

Sub-domain	k_x	k_y
$(0,0.5] \times (0,0.5]$	1	1
$(0.5,1) \times (0,0.5]$	100	1
$(0,0.5] \times (0.5,1)$	1	100
$(0.5,1) \times (0.5,1)$	100	100

Example DM4: (Anisotropic coefficient) 5-point finite difference. $k_x = 100$, $k_y = 1$, $\sigma = 0$.
The numbers of iterations for these examples are shown in Table 1, which also gives equivalence, compatibility and consistency [26]. Examples DM1 and DM4 correspond to the analytical results

of this paper.

In Example DM1, the number of iterations for fully compatible orderings is in the range of 20-24, whereas it is more than 25 for partially compatible or incompatible orderings. More specifically, the *rb* and *altd* require 38 iterations, 65% more than the *row* ordering. Intuitively speaking, then, the greater the number of incompatible nodes exist, the poorer the convergence is. A typical example of a low number of incompatible nodes is the *loc* (incompatible) ordering, which has only four (of 100 nodes total). Its number of iterations is the same as that of the *row* ordering.

In Example DM4 (the anisotropic case), the differences in convergence are striking. Each of the fully compatible and partially x compatible orderings requires only 9 iterations, whereas the others require more than 17. The *mind* requires 48 iterations and the *rb* and *altd* require 47 iterations. The poor convergence of *mind*, *rb* and *altd* orderings reflects their dense distribution of incompatible nodes. Among incompatible orderings, the *loc* requires the least iterations. This is because it has the least incompatible nodes.

Figures 10–12 suggest that strong anisotropy in parameters improves the convergence of compatible orderings, whereas it deteriorates the convergence of incompatible orderings. Examples DM1 and DM4 agree well with this view.

Table 1 also indicates consistency in ordering [26], an important guideline for SOR methods. We may see that a good ordering for the SOR is not always a good ordering for the ILU. For example, *rb* ordering, consistent but not compatible, is poor. *Spiral* ordering, fully compatible but not consistent, is competitive with *row* ordering.

4.2. Authors' experiments. We have implemented here, as preconditioners for the conjugate gradient squared procedure, ILU factorizations based on Row, VDV, Zbx, NZx, Zby and RB orderings (Fig. 1) [20]. An advantage of the CGS method over the Bi-conjugate Gradient [11] is that the CGS does not require M^T.

The model equations are of the form

$$(k_x u'_x)'_x + (k_y u'_y)'_y + v_x u'_x + v_y u'_y = g \quad \text{on } \Omega = (0,1) \times (0,1).$$

Here, the advection and diffusion terms are approximated by the mixed 1st-order upwind and 2nd-order central differencings, and the 2nd-order central differencing, respectively. The mixed differencing is arranged in such a way that it becomes equal to the centered difference when the Cell-Peclet number, e.g., $\Delta x|v_x|/k_x$, is small, and that it becomes equal to the full upwind difference when it is large, so as to keep the diagonal dominance of the matrix.

The following tests were made. In Example DL1, the effect of diffusion parameter anisotropy was examined where $v_x = v_y = 0$. In Example DL2, the effect of convection term intensity was tested. In Example DL3, the effect of non-uniform grids was examined. Non-uniform grids were treated by mapping (x, y) to (ξ, η) as follows:

$$k_x(\xi'_x)^2 u''_{\xi\xi} + k_y(\eta'_y)^2 u''_{\eta\eta} + (k_x \xi''_{xx} + v_x \xi'_x) u'_\xi + (k_y \eta''_{yy} + v_y \eta'_y) u'_\eta = 0.$$

Here, $h' = \Delta\xi = \Delta\eta = 1$, and $x = x(\xi)$ and $y = y(\eta)$ are given numerically by $x_i = x(i-1)$ and $y_j = y(j-1)$. Hence, non-uniform grids affects convection term intensity.

Example DL1: $v_x = v_y = 0$, $g = 0$, $k_x/k_y = 1/128, \ldots, 128/1$.

B.Cs.: $u|_{x=0} = u|_{y=0} = 0$, $u|_{x=1} = u|_{y=1} = 100$.

Example DL2: $k_x = k_y = 1$, $g = 0$.

B.Cs.: $u|_{x=0} = u|_{y=0} = 0$, $u|_{x=1} = u|_{y=1} = 100$.

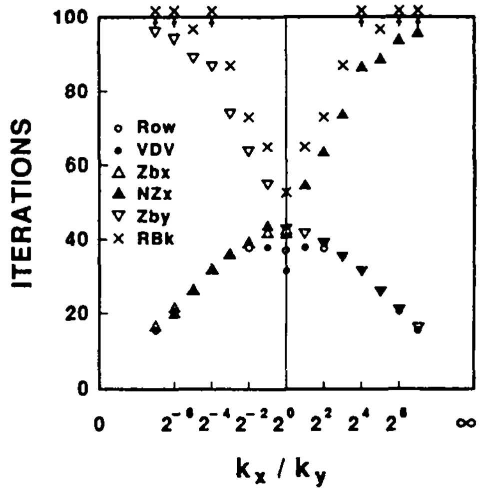

FIG. 13. *Numerical results. - Example DL1.*

Case 1: $v_x = 0, \ldots, 256$, $v_y = 0$(fixed).
Case 2: $v_y = 0, \ldots, 256$, $v_x = 0$(fixed).

Example DL3: $k_x = k_y = 1$, $v_x = v_y = 0$, $g = 1$. B.C.: $u = 0$ on $\partial\Omega$.
Case 1: $\alpha = 1.0, \ldots, 1.5$, $\beta = 1.0$(fixed).
Case 2: $\alpha = \beta = 1.0, \ldots, 1.5$.

$$\alpha = \begin{cases} \Delta x_i / \Delta x_{i-1} & \text{if } i \le n_x/2, \\ \Delta x_i / \Delta x_{i+1} & \text{if } i > n_x/2. \end{cases}$$

$$\beta = \begin{cases} \Delta y_j / \Delta y_{j-1} & \text{if } j \le n_y/2, \\ \Delta y_j / \Delta y_{j+1} & \text{if } j > n_y/2. \end{cases}$$

Domain Ω was discretized into a 51×51 grid. Iterations were terminated when $||r||_2/||b||_2 \le 10^{-6}$ or when they reached 100. The starting vector $u^{(0)}$ was $\text{diag}^{-1}(A)b$. Figures 13–15 show numerical results. Their vertical axes correspond to number of iterations and their horizontal axes to variable parameters. Arrows pointing to entries outside the graphs indicate lack of convergence within 100 iterations.

Overall, the convergence of Row and VDV fully compatible orderings tends to improve as anisotropy or convection intensity increases. On the other hand, the convergence of RB incompatible ordering degrades as either one of them increases. Zbx, NZx and Zby partially compatible orderings behave like fully compatible orderings when the direction of anisotropy or convection intensity coincides with the direction of compatibility. Moreover, a good coincidence is observed between the size of dominant remainder terms and the number of iterations when the anisotropy of the diffusion parameters move above or below 1 (see Figs. 10, 11, 12 (a) and 13). When the intensity of the convection parameters move above or below 0, coincidence is not so clear (see Figs. 10, 11, 12 (b), (c) and 14). When $k_x/k_y \simeq 8$, partially y compatible and incompatible orderings require twice as many iterations as the fully or partially x compatible orderings. When k_x/k_y reaches 16, RB ordering fails to converge. The same observation can also be made for incompatible and partially compatible orderings in Examples DL2 and DL3.

Since non-uniform grids often appear in real computation, e.g., CFD and electronics device simulation, robustness of convergence versus change in these parameters is important. In this

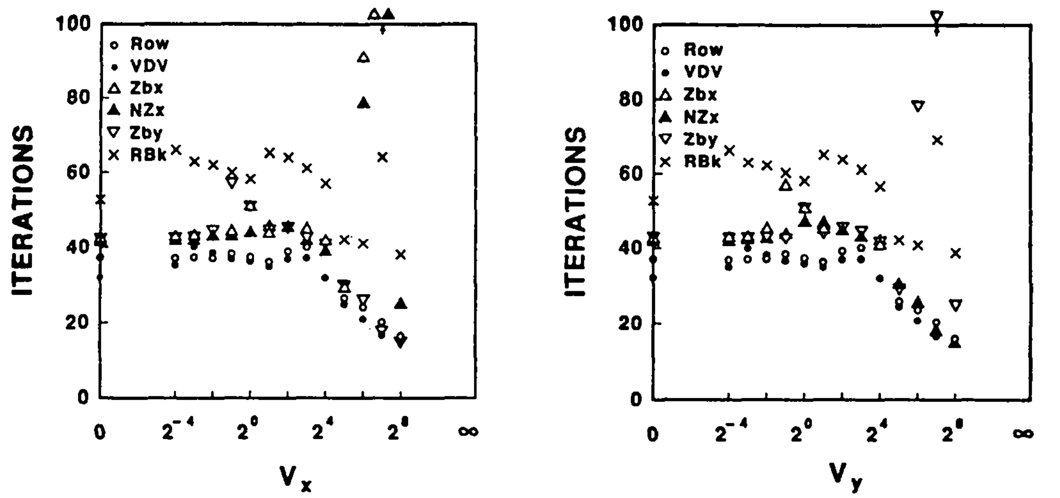

FIG. 14. *Numerical results. - Example DL2.*

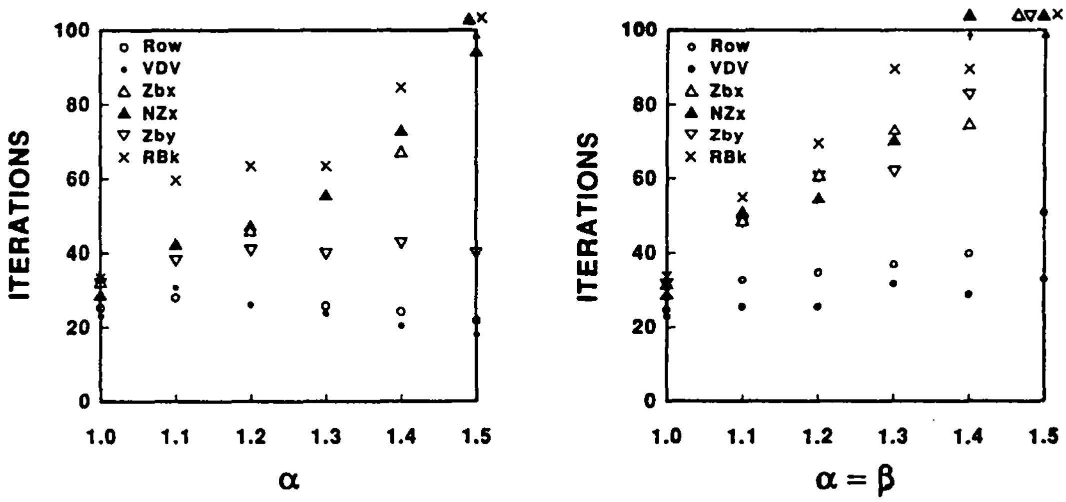

FIG. 15. *Numerical results. - Example DL3.*

respect, Row and VDV fully compatible orderings are to be recommended. In terms of parallelism (§2.3), VDV ordering is preferable. Zbx, NZx and Zby partially compatible orderings can perform almost as well as fully compatible ordering when they are carefully mapped to the domain.

5. Conclusion. We have conducted here analytical and empirical studies regarding the effects of ordering on the speed of convergence in ILU preconditioned iterative methods. Full compatibility and partial compatibility of ordering were defined in terms of the digraph structure for ordering. Analytical and empirical results have shown that these properties can fairly well explain the effects of ordering on convergence in ILU preconditioned methods.

Parallelism is another requirement for efficient numerical algorithms. VDV ordering offers the highest degree of parallelism among fully compatible orderings, and a higher degree yet can be attained with partially compatible orderings. Zebra and ND-Zebra orderings are in this category, but studies need to be made regarding the implementation of these orderings in parallel and/or vector computers (e.g., see [4]). Another possible approach is the use of incompatible ordering which has a low number of incompatible nodes (e.g., see [5]).

The analytical results of this paper for the 2D 5-point finite difference case can be extended to the 3D 7-point finite difference case, and generalizations of the results for higher-order finite difference and finite element cases are important subjects for further study.

Acknowledgments. S. Doi would like to thank Henk Van der Vorst for bringing his attention to the graph aspect of ordering, and Gene Golub for his constructive and educational comments and suggestions.

REFERENCES

[1] C. C. Ashcraft and R. G. Grimes, *On Vectorizing Incomplete Factorization and SSOR Preconditioners*, SIAM J. Sci. Stat. Comput., 9(1988), pp. 122–151.

[2] E. F. D'Azevedo, P. A. Forsyth and W. P. Tang, *Ordering Methods for Preconditioned Conjugate Gradient Methods Applied to Unstructured Grid Problems*, Proc. Copper Mountain Conference on Iterative Methods, 1990.

[3] S. Doi and N. Harada, *Tridiagonal Factorization Algorithm: A Preconditioner for Nonsymmetric System Solving on Vectorcomputers*, J. Inf. Process., 11(1988), pp. 38–46.

[4] S. Doi and A. Lichnewsky, *Some Parallel and Vector Implementations of Preconditioned Iterative Methods on Cray-2*, Int. J. High Speed Comput, 2(1990), pp. 143–179.

[5] S. Doi, *On Parallelism and Convergence of Incomplete LU Factorizations*, Appl. Numer. Math., 1990, to appear.

[6] I. S. Duff and G. A. Meurant, *The Effect of Ordering on Preconditioned Conjugate Gradients*, BIT, 29(1989), pp. 635–657.

[7] I. S. Duff, A. M. Erisman and J. K. Reid, *Direct Methods for Sparse Matrices*, Oxford University Press, Oxford, 1989.

[8] V. Eijkhout, *Vectorizable and Parallelizable Preconditioners for the Conjugate Gradient Methods*, PhD Thesis, Catholic University of Nijmegen, 1990.

[9] H. C. Elman and E. Agrón, *Ordering Techniques for the Preconditioned Conjugate Gradient Method on Parallel Computers*, Comput. Phys. Comm., 53(1989), pp. 253–269.

[10] J. Erhel, A. Lichnewsky and F. Thomasset, *Some Algorithms for Vector or Parallel Computers*, Physica, 124A(1983), pp. 587–602.

[11] R. Fletcher, *Conjugate Gradient Methods for Indefinite Systems*, Lecture Note in Mathematics, No.506, Springer-Verlag, 1976, pp. 73–89.

[12] A. George, *Nested Dissection of a Regular Finite-Element Mesh*, SIAM J. Numer. Anal., 10(1973), pp. 345–363.

[13] F. Harary, *Graph Theory*, Addison-Wesley Publishing Company, Reading, 1969.

[14] D. KERSHAW, *The Incomplete Cholesky-Conjugate Gradient Method for the Iterative Solution of Systems of Linear Equations*, J. Comput. Phys., 26(1978), pp. 43–65.

[15] A. LICHNEWSKY, *Some Vector and Parallel Implementations for Preconditioned Conjugate Gradient Algorithms*, NATO ASI, F7(1984), pp. 343–359, Springer-Verlag.

[16] J. A. MEIJERINK AND H. A. VAN DER VORST, *An Iterative Solution Method for Linear Systems of Which the Coefficient Matrix is a Symmetric M-Matrix*, Math. Comput., 31(1977), pp. 148–162.

[17] J. M. ORTEGA, *Orderings for Conjugate Gradient Preconditioners*, Computer Science Report, TR-90-24, 1990.

[18] Y. OYANAGI, *Hyperplane vs. Multicolor Vectorization of Incomplete LU Processing for Wilson Fermion on the Lattice*, J. Inf. Process., 11(1987), pp. 32–37.

[19] E. L. POOLE AND J. M. ORTEGA, *Multicolor ICCG Methods for Vector Computers*, SIAM J. Numer. Anal., 24(1987), pp. 1394–1418.

[20] P. SONNEVELD, *CGS, A Fast Lanczos-Type Solver for Nonsymmetric Linear Systems*, SIAM J. Sci. Stat. Comput., 10(1989), pp. 36–52.

[21] H. A. VAN DER VORST, *Iterative Solution Methods for Certain Sparse Linear Systems with a Non-Symmetric Matrix Arising from PDE-Problems*, J. Comput. Phys., 44(1981), pp. 1–19.

[22] H. A. VAN DER VORST, *Preconditioning by Incomplete Decompositions*, PhD Thesis, Delft University of Technology, 1982.

[23] H. A. VAN DER VORST, *Large Tridiagonal and Block Tridiagonal Linear Systems on Vector and Parallel Computers*, Parallel Comput., 5(1987), pp. 45–54.

[24] H. A. VAN DER VORST, *Bi-CGSTAB: A Fast and Smoothly Converging Variant of Bi-CG for the Solution of Nonsymmetric Linear Systems*, SIAM J. Sci. Stat. Comput., 1990, submitted.

[25] R. S. VARGA, *Matrix Iterative Analysis*, Prentice-Hall, Englewood Cliffs, New Jersey, 1962.

[26] D. M. YOUNG, *Iterative Solution of Large Linear Systems*, Academic Press, New York, 1971.

Imprimé en France

par

l'Institut National de Recherche en Informatique et en Automatique.

ISSN 0249-6399